Reading for Lectures 34-35: PKT Chapter 12.
Will try to return Problem Set 5 Friday.
Will try for Monday?: new data sheet and draft formula sheet for final exam.
Did Couette cell demonstration of non-mixing hydrodynamics

Our starting point for hydrodynamics are two equations:

Continuity equation: $\vec{\nabla}\cdot\vec{v} = 0$

Navier-Stokes equation: $\rho\dfrac{\partial\vec{v}}{\partial t} + \rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}P + \eta\nabla^2\vec{v}$ (g is set to zero, see Lect. 34)

Before starting, I want to do a dimensional analysis of the Navier-Stokes equation:

Dimensions: $[\rho] = \left[\dfrac{M}{L^3}\right]$; $[v] = \left[\dfrac{L}{T}\right]$; $[\eta] = \dfrac{[F\cdot T]}{[L^2]} = \dfrac{[M]}{[LT]}$, (since $F = \eta\dfrac{Av}{d}$)

plus one or more length scales set by system geometry. (Assumes no time-dependent driving.)
Assume one such scale; call it ℓ, $\left([\ell] = [L]\right)$. e.g., the radius R of the sphere or the pipe.

The scales of P and v are generally set by boundary conditions (e.g., examples above) and they are NOT independent. We choose to work with v, treating P as dependent.

Without the geometry scale, we have v, ρ, and η, from which we can construct natural scales:

length: $[L] = \dfrac{\eta}{\rho v}$

time: $[T] = \dfrac{\eta}{\rho v^2}$

mass: $[M] = \dfrac{\eta^3}{\rho^2 v^3}$, from which anything else can be constructed.

e.g., force: $[F] = \dfrac{\eta^2}{\rho}$; pressure: $[P] = \rho v^2$

Note: If the non-linear ("inertial") term is absent, then ρ drops out of the equations, so none of these quantities has a natural scale. We will see what happens with this in the pipe-flow problem.

As soon as you add an additional length scale ℓ, then for the first time there is a dimensionless ratio,

$$\frac{\ell}{[L]} = R_e \equiv \frac{\rho v \ell}{\eta}.$$ **Reynolds number**

Such dimensionless numbers have, as we shall see, a profound significance.

Note that R_e measures the ratio of the inertial term $\rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} \sim \dfrac{\rho v^2}{L}$ to the viscous force $\eta\nabla^2\vec{v} \sim \dfrac{\eta v}{L^2}$ when/where the characteristic scale of variation of the velocity field is ℓ:

$$\frac{\text{inertial term}}{\text{viscous term}} = \frac{\rho v^2/L}{\eta v/L^2} = \frac{\rho v L}{\eta} \to \frac{\rho v \ell}{\eta} = R_e.$$

Thus, when/where Reynolds number is large, viscous dissipation is small, and motion is dominated by inertial flow, which easily produces vorticity/eddies and related irregular/turbulent flow.
When/where Reynolds number is small, viscous dissipation is dominant, inertial effects are weak, potential eddies dissipate via viscous friction before they develop, and motion remains laminar.

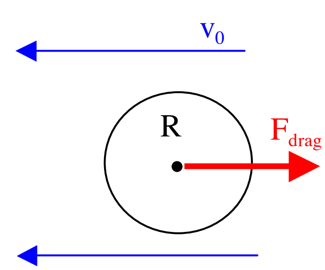
How big is Reynolds number in examples:
(a) Macroscopic hydrodynamics (you swimming):
$\rho = 10^3 \text{ kg/m}^3; v = 1 \text{ m/s}; \eta = 10^{-3} \text{ Pa s}; \ell = 1 \text{ m}$, so $R_e = 10^6$.
(b) Cellular hydrodynamics (bacterium swimming):
$\rho = 10^3 \text{ kg/m}^3; v = 10^{-6} \text{ m/s}; \eta = 10^{-3} \text{ Pa s}; \ell = 10^{-6} \text{ m}$, so $R_e = 10^{-6}$.
Comment: glycerine has viscosity about 1 Pa s; molasses ~ 5 Pa s; tar ~ $3\text{x}10^7$ Pa s.
If you want to imagine what it is like for a bacterium in water, imagine swimming is a fluid with a viscosity 30 times that of tar! (this is from the point of view of the smoothness of the flow, not the forces involved!)

Rule of thumb: When $R_e > R_{critical} = 1000—2000$, turbulence is unavoidable.
$R_{critical}$ is geometry-dependent.

What does this mean in terms of solving for steady-state flows?
You can look for and find solutions of the steady-state equations for both high and low Reynolds number (see below); but, at high Reynolds number, these solutions are generally completely unstable in the sense that any infinitesimal disturbance will grow exponentially in time and lead to turbulent flow.
Comment: This is the opposite of the diffusion equation, where you found that disturbances away from equilibrium and steady states all decay exponentially in time.

Upshot:
1. When/where Reynolds number is small, you can drop inertial term, so
$\rho \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} P + \eta \nabla^2 \vec{v}$ or for the steady state $0 = -\vec{\nabla} P + \eta \nabla^2 \vec{v}$.
Note similarity to diffusion equation.
Note that the fluid density ρ disappears from the problem in this limit.
This is a linear equation and is relatively simple to solve.
2. When/where the Reynolds number is large, you can drop the viscous term, so
$$\rho \frac{\partial \vec{v}}{\partial t} + \rho\left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} = -\vec{\nabla} P \text{ or for the steady state } \rho\left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} = -\vec{\nabla} P.$$
This is a nonlinear equation. The limit $\eta \to 0$ is technically singular ("singular perturbation"). Small η is difficult. Flow is turbulent and calculations are usually difficult/impossible.

Some Examples: (draw flow lines)



1. Stokes drag (low Reynolds number) (scaling analysis only)
What force does it take to move a sphere of radius R through an incompressible fluid with velocity v_0?
Result: $\boxed{F_{drag} = 6\pi\eta R v_0}$: Stokes law
Geometry: Uniform bulk flow, fixed sphere of radius R.
Look for steady-state flow:
$\vec{v}(\vec{r}, v_0, R, \eta) = v_0 \vec{f}\left(\frac{\vec{r}}{R}, R_e = \frac{\rho v_0 R}{\eta}\right)$ by dimensional analysis.
But, because ρ must disappear in the low-Reynolds number limit, we expect
$\vec{v}(\vec{r}, v_0, R, \eta) = v_0 \left[\vec{f}\left(\frac{\vec{r}}{R}\right) + \frac{\rho v_0 R}{\eta} \vec{g}\left(\frac{\vec{r}}{R}\right)\right]$.

Similarly, $F_{drag} = \frac{\eta^2}{\rho} f\left(R_e = \frac{\rho R v_0}{\eta}\right)$, which must at low Reynolds number go linearly in R_e **35.3**

(to cancel the ρ dependence); hence, $F_{drag} = \frac{\eta^2}{\rho} \cdot C \frac{\rho R v_0}{\eta}\left[1 + O\left(\frac{\rho R v_0}{\eta}\right)\right] = C\eta R v_0 \left[1 + O\left(\frac{\rho R v_0}{\eta}\right)\right]$.

The exact result is C=6π and $F_{drag} = 6\pi\eta R v_0 \left[1 + \frac{3}{8}\frac{\rho v_0 R}{\eta} + O\left(R_e^2\right)\right]$.

For macroscopic objects the v^2 term is much larger than Stokes and dominates.

Technical note: The formula for the velocity field fails at long distance even when R_e is small, since, as the perturbations due to the sphere die out, eventually L (scale of variation of the velocity) becomes large and the inertial term can no longer be neglected.

This looks a bit like magic but it is easy to see by simple qualitative arguments:

(a) In the low Reynolds limit, the only variables present are $[v] = \left[\frac{L}{T}\right], [\eta] = \frac{[FT]}{[L^2]}$, and $[R] = L$, so the only way to construct a quantity with dimensions of force is $[F] = [\eta R v_0]$.

(b) In the low-Reynolds limit, ρ is absent, so there is no natural length scale other than R. The velocity field must vanish at the surface of the sphere (no-slip boundary condition). It follows that the velocity field varies on the scale of R around the sphere and, thus, the velocity shear goes as $\frac{dv}{dz} \sim \frac{v_0}{R}$. Thus, estimate drag on, say, cylinder of radius R and length 2R:

$F_{drag} = \eta A \frac{dv}{dz} \sim \eta(2\pi R \cdot 2R)\frac{v_0}{R} = 4\pi\eta R v_0$. (vs sphere result of $F_{drag} = 6\pi\eta R v_0$.)

(Of course, the 4π factor should not be taken seriously!)

Application: Sedimentation and centrifugation:

Consider a spherical mass of radius R and density ρ_m (> ρ_w) released from rest in water.

$F_T = -(m - m_w)g - \gamma v = ma = m\frac{dv}{dt}$, with Stokes-law friction coefficient $\gamma = 6\pi R\eta$.

After initial acceleration, sphere approaches a constant terminal downward velocity $v_{term} = -v_{drift}$ with $v_d = \frac{(m - m_w)g}{\gamma} = \frac{4\pi R^3(\rho_m - \rho_w)g}{3(6\pi R\eta)} = \frac{2}{9}\frac{R^2(\rho_m - \rho_w)g}{\eta}$.

Approach to this limit is easy to calculate:

Equation of motion can be rewritten $\frac{dv}{dt} + \frac{v}{\tau} = -\frac{v_d}{\tau}$ with $\frac{1}{\tau} = \frac{\gamma}{m} = \frac{6\pi R\eta}{m}$.

This has the immediate solution (bc: v=0 at t=0): $v(t) = -v_d + v_d e^{-t/\tau}$, i.e., exponential approach to the terminal velocity $-v_d$.

Estimate quantities for micron-sized object with density of rock, say, 5x water, i.e. specific gravity=5.

Suppose sphere of diameter= 1 μm.

I find $v_d = 2.2 \times 10^{-12}\ m/s$ and $\tau = 2.8 \times 10^{-7}\ s$.

Messages:

- v_d is very slow: 5×10^9 s to travel 1 cm (3×10^7 s/yr).
  small particles remain suspended for a long time.
  use of centrifuges necessary for sedimentation, effective $g_{eff}=10^6$ g.
- Time to get to terminal velocity is negligible.